

# Network Debt

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## Abstract

Given a financial network of liabilities, we consider the following question: *which agent's potential default would implicate the most number of agents in the network structure?* Simply comparing each agent's total debt completely ignores the structure of the network. For propagation effects of a potential default, owing to a creditor who does not owe anybody and owing to a creditor who is heavily in debt himself clearly do not have the same implications even when the amount owed is the same. The liability network structure must be taken into account in order to capture downstream effects. We propose a general notion which addresses this issue quantitatively. Our notion yields a ranking of all agents in the network in terms of severity of default implications. Potential regulatory applications include stress testing for banks.

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# 1 Introduction

The issue of contagion in financial networks has been the subject of much recent attention. In the seminal paper Allen and Gale (2000), the classical two-period Diamond-Dybvig banking model was extended to feature four regions with idiosyncratic liquidity demands correlated in a specified way. The presence of idiosyncratic shocks prevents the representative banks in each region from offering consumers the social planner’s first-best contract. This can be circumvented by allowing inter-bank deposits before the realization of shocks, for the specified correlation of liquidity demand shocks. Once the four representative banks achieve an inter-bank deposit configuration that implements the first-best contract, one bank is then subjected to an unforeseen liquidity demand. After comparing three network configurations, Allen and Gale conjectured that the more connected networks can better survive intact from the spill-over effect of a failed bank. More connected configurations means when a failed bank is forced to liquidate its deposits, this shock is shared by a larger number of neighbors. More recently, Acemoglu *et al.* (2010) constructs a three-period banking model. Banks make investment decisions and form endogenously a liability network in first period. Inter-bank deposit is again used as a device by banks to hedge against liquidity demands. A stochastic short-term return is realized in the second period. A bank experiencing negative shock may not be able to meet its obligations and default. Assuming  $m$  out of  $n$  banks suffer negative shocks, a financial network is judged based on two criterion— resilience (worst case) and stability (average case). The analysis focuses on regular networks, where each bank has the same number of creditors and debtors—in particular, the complete and ring networks. They found such networks exhibit “phase-transition” between the small shock and big shock regimes. In the small shock regime, the ring network is the least resilient/stable while the complete network is the most resilient/stable. This comports results from the Allen and Gale model, with similar reasoning. However, when shock is large, the default mechanism specified by the model means connectedness allows a default to transmit through the entire network, making both the complete and ring networks both least resilient/stable. The more stable networks in this regime are those consisting of weakly connected components and thus able to isolate the propagation of default to a single component.

In the above literature, networks are built-in features of a model. Just as the model prescribes specific rules for agent behavior, model-specific configurations are also imposed on the network. While this allows one to extract economic insight in the context of the model, the scope of network analysis is limited to the confines of the model. The types of networks for which meaningful statements can be made are often restricted to stylized cases.

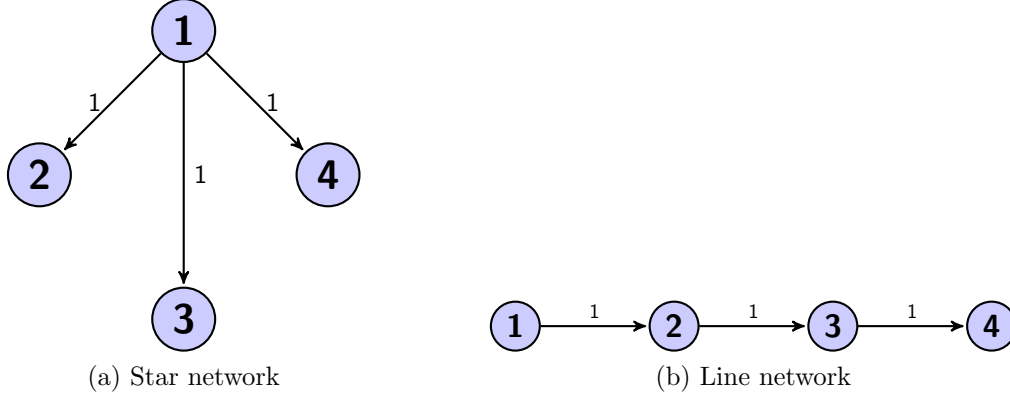


Figure 1: Same amount of total liability can mean different default implications

Therefore the passage from economic insights to concrete recommendations for the regulator, who is interested in the very same question—the propagation effects of defaults on the network—is not immediately clear. To address this, we take a complementary perspective and ask a more general question: given an arbitrary financial network of liabilities,

*How does one compare the agents’ potential defaults in terms of the severity of implications for the network structure?*

As existing literature already indicates, simply comparing agents’ total liabilities is not satisfactory, even for the simplest network structures. For example, consider the networks depicted in Figures 1(a) and 1(b). In both networks, agent 4 carries one unit of debt. However, assuming there is no inflow of capital exogenous to the network, the implications of agent 4 defaulting are dramatically different for the two networks. In the star network of Figure 1(a), agent 4’s default is an isolated event. The rest of the network remains intact. On the other hand, in the line network of Figure 1(b), one unit of asset is passed through the agents with agent 4 being the final debtor. The default of agent 4 would propagate through the entire network and cause it to disintegrate. Therefore, any satisfactory answer must take the network structure into account and should be required to satisfy certain natural properties. For such a “risk indicator”  $i \mapsto d_i$  that assigns agent  $i$  a “riskiness weight”  $d_i$ , some minimum requirements may be axiomized as follows:

- A1.** If agent  $i$ ’s total liability is zero,  $d_i$  is zero.
- A2.**  $d_i$  is monotone non-decreasing with respect to agent  $i$ ’s liability.
- A3.** If  $j$  is a creditor of  $i$ ,  $d_i$  is monotone non-decreasing with respect to  $d_j$ .

Axiom A1 is certainly reasonable. An agent who owes nothing has no default implications. Axiom A2 requires  $d_i$  to account for agent  $i$ ’s own liabilities. Per Axiom 3, if a creditor  $j$

of  $i$  becomes more leveraged,  $d_i$  increases. Thus Axiom 3 means increase in  $d_j$  is reflected downstream by  $d_i$ . Owing to a more leveraged creditor should lead to more severe network implications in the event of default. In addition, for applicability, any proposed  $d_i$  should be easy to compute. Such an answer would give the regulator a device that helps identify network vulnerabilities, e.g. stress-testing for banks.

In Section 2, a candidate notion of  $d_i$ , which we call *network debt*, is proposed. We start with a simple expression that is a weighted sum of an agent's liabilities. On an informal level, this expression captures the intended characteristics of the network structure: the weight of a particular debt is determined by the indebtedness of the creditor to whom it is owed. Debts owed to creditors who owe larger amounts receive higher weight than debts owed to creditors who have relatively less debt themselves. We then impose further specifications to make this well-defined and arrive at an eigenvector of the adjacency matrix. While technical at first glance, the specifications have certain graph-theoretical underpinnings. We also point out that similar objects have been encountered before in economic contexts. In Section 3, with a candidate notion in hand, we check it against some stylized networks as a first test and find that it behaves as intended. The calculations done for the stylized examples are in fact general properties of our notion. Section 4 states theorems showing that, network debt satisfies Axioms A1 - A3, and other reasonable properties. The proofs are relegated to the Appendix.

## 2 Definition

Consider a network in which nodes  $\{1, \dots, n\}$  are financial agents (e.g. banks) and each directed edge from  $i$  to  $j$  has weight  $a_{ij} \geq 0$ , the amount agent  $i$  owes agent  $j$ . For example, in an inter-bank network, columns and rows of the matrix  $A = [a_{ij}]$  specify the inter-bank assets and liabilities of a given bank, respectively. In this context, the diagonal entries  $a_{ii}$ ,  $i = 1, \dots, n$ , are zero.

For having a notion of debt which accounts for the network structure, one can envision attaching a weight  $d_i$  to agent  $i$ , for  $i = 1, \dots, n$  such that  $d_i$  is a weighted sum of agent  $i$ 's liabilities  $\{a_{ij}, j = 1, \dots, n\}$  where the weights  $d_j$  themselves take into account the indebtedness of creditor  $j$ :

$$d_i = \sum_{j=1}^n a_{ij} d_j, \quad 1 \leq i \leq n.$$

Should such  $d_j$ 's exist, Axiom A1 holds immediately. While non-trivial to establish, it is not unreasonable to conjecture at this point that Axioms A2 and A3 may hold. If agent  $i$ 's liabilities  $\{a_{ij}, j = 1, \dots, n, j \neq i\}$  increase on the right hand side,  $d_i$  should increase. Similarly, an increase in  $\{d_j, j = 1, \dots, n, j \neq i\}$  should cause an increase in  $d_i$ .

The above expression of  $d_i$ ,  $1 \leq i \leq n$ , means that the vector  $(d_i)$  is an eigenvector of the adjacency matrix  $A$  corresponding to eigenvalue 1. 1 may not be an eigenvalue of  $A$ . However, eigenvectors are invariant under scaling. In other words, the proposed  $d_i$ 's capture the *relative* severity of defaults,  $\{\frac{d_i}{d_j}\}_{i \neq j}$ , of the agents in the network. So without losing any information we can allow for an overall scaling factor  $\lambda$  which ensures the existence of  $d_i$ 's. Thus one arrives at the equation

$$d_i = \lambda \sum_{j=1}^n a_{ij} d_j.$$

This characterizes the vector  $(d_i)$  as an eigenvector of the adjacency matrix  $A = [a_{ij}]$  of the network corresponding to eigenvalue  $\lambda$ .

**Definition 1.** *Given a strongly-connected financial network of  $n$  agents with non-negative edge weights  $\{a_{ij}, 1 \leq i, j \leq n\}$ , the (relative) network debt distribution is an eigenvector  $d$  of  $A = [a_{ij}]$  corresponding to the positive eigenvalue  $\lambda$  with maximum modulus<sup>1</sup> and normalized so that  $\sum_{i=1}^n d_i = 1$ . The  $i$ -th entry  $d_i$  of  $d$  is the (relative) network debt of agent  $i$ .*

At this initial stage, we remark that the maximum modulus condition on the eigenvalue  $\lambda$  make network debt well-defined and also positive. That is, the vector  $d$  always has positive entries. Although seemingly technical at first glance, the maximal eigenvalue  $\lambda$  is known to reflect certain properties of the underlying graph (see, for example, Chung (1997)). The properties  $\lambda$  with respect to perturbations of  $A$  are key in establishing the theorems in Section 4. Any financial liability network (with non-negative edge weights) can be made strongly connected by an arbitrarily small perturbation of the adjacency matrix<sup>2</sup>. So the network debt of an arbitrary network can be defined by a limiting procedure.

**Definition 2.** *Given a general financial network of  $n$  agents with non-negative edge weights  $\{a_{ij}, 1 \leq i, j \leq n\}$ , the (relative) network debt  $(d_i)$  of the network is the limit*

<sup>1</sup>The modulus of a complex number is its length as a vector in  $\mathbb{R}^2$ .

<sup>2</sup>More formally, we identify a network with its adjacency matrix. Under this identification, strongly connected networks are dense in the class of networks under consideration, in the unique locally convex topology on matrices.

$$\lim_{\eta \rightarrow 0^+} (d(\eta)_i)$$

where  $(d(\eta)_i)$  is the network debt of  $A + \eta J$  for some strongly connected  $J$ .

The technical details and a more precise discussion on the definition are postponed to Section 4. As a first step towards showing network debt captures the the relative severity of an agent’s default, we consider in Section 3 some stylized examples for which the network debt can be readily computed.

## 2.1 Similar objects in other economic contexts

Objects similar to the eigenvector  $d$  have been encountered before in other economic contexts. In the special case where  $A$  is symmetric,  $d$  is proportional to the vector

$$(A + \frac{1}{\lambda}A^2 + \frac{1}{\lambda^2}A^3 + \dots)\mathbf{1},$$

where  $\mathbf{1}$  denotes the vector whose entries are all 1’s. In the present context of financial networks, the  $n$ -th term in the series is agents’ “ $n$ -th degree debt” discounted by the factor  $\frac{1}{\lambda^n}$ . The series converges to<sup>3</sup>

$$(I - \frac{1}{\lambda}A)^{-1}A\mathbf{1}.$$

On the other hand, Acemoglu *et al.* (2012) studies the *influence vector* of a multi-sector production-trade network. If  $A$  is the  $n \times n$  matrix of factor shares in a  $n$ -sector economy with Cobb-Douglas technologies with total factor share  $1 - \alpha$ , the *Leontief inverse* is  $(I - (1 - \alpha)A)^{-1}$  and the influence vector is

$$v = \frac{\alpha}{n}(I - (1 - \alpha)A)^{-1}\mathbf{1}.$$

Whereas the network debt (in the symmetric case) is proportional to an agent’s total discounted downstream debts, the influence vector measures how far an idiosyncratic shock of productivity in one sector would propagate downstream.

If  $A$  is the adjacency matrix of the network in the multi-region banking model of Allen and Gale (2000), and  $A_{11}$  is the adjacency matrix of the sub-network consisting of those

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<sup>3</sup>Choosing  $\lambda$  to be the maximal eigenvalue guarantees that the series converges.

banks who defaulted<sup>4</sup>, then the liquidation value of assets of the failed banks is given by

$$v = (I - \frac{1}{2}A_{11})^{-1}a_o$$

where vector  $a_o$  specifies a failed bank's inter-bank asset external to the failed sub-network. The discount factor is  $\frac{1}{2}$  because the model exogenously specify the inter-bank liability is half of a bank's total liability; the other half is owed to the consumer. Our notion of network debt focuses only on liabilities and sums up an agent's debts downstream by multiplying by the vector  $\mathbf{1}$  whereas, in the Allen-Gale banking model, the Leontief inverse is multiplied by endogenous assets gives the liquidation value.

### 3 First examples

The properties of network debt reflected in the stylized examples of this section are in fact general properties, which will be shown in Section 4.

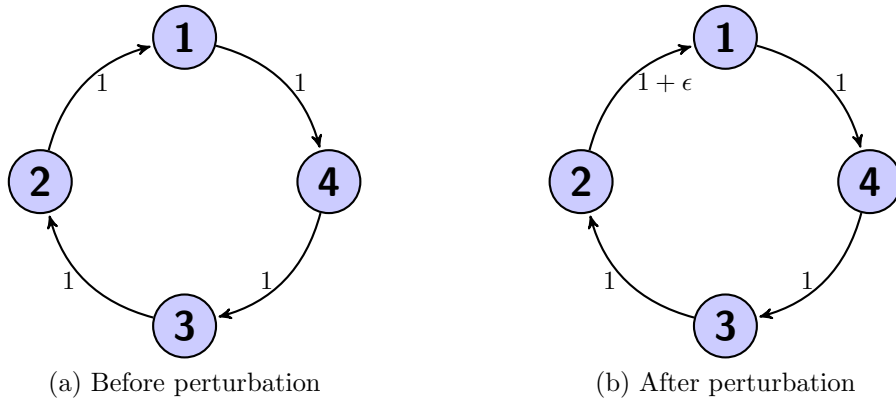


Figure 2: A ring network where one unit of asset circulates among 4 agents

**Example 3.1.** *Figure 2(a) depicts a ring network where one unit of asset circulates among the agents. As any agent's default causes the network to disintegrate, the network debt should indicate equal severity in this case. The adjacency matrix is*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

<sup>4</sup>The matrix  $A_{11}$  is a principal sub-matrix of  $A$ .

This gives network debt

$$\begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

with corresponding to maximal eigenvalue  $\lambda = 1$ . In this configuration, each agent carries the same network debt.

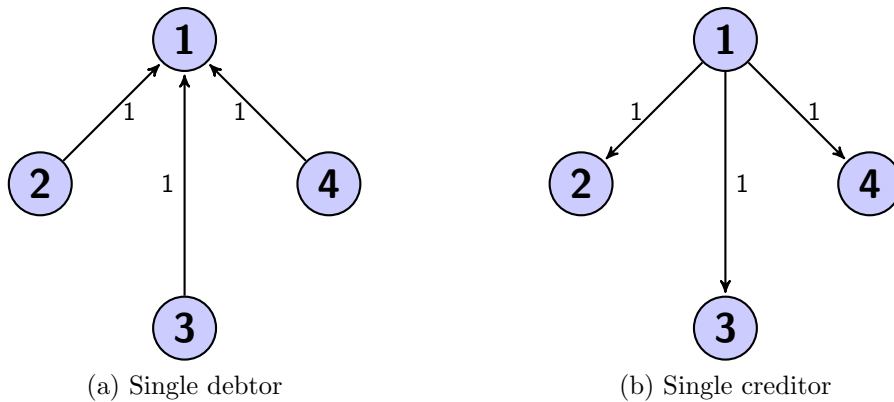


Figure 3: A star network and its reverse

**Example 3.2.** Figure 3(a) depicts the star network, where there is a single debtor in the network, agent 1. Agents 2, 3, and 4 cannot default while the integrity of the entire network hinges on agent 1 honoring his liabilities. The adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The network debt is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Agent 1 carries an infinite amount of network debt relative to all the other agents. Reversing the edges, shown in Figure 3(b), makes agent 1 the sole creditor of the network with resulting

adjacency matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and network debt

$$\begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

In this reversed configuration, agent 1 cannot default while any default by agents 2, 3, or 4 destroys one-third of the network.

**Remark 3.3.** *The maximal eigenvalues for these two star networks are both 0. These networks are not strongly-connected. For these networks, one needs to apply the more general Definition 2.*

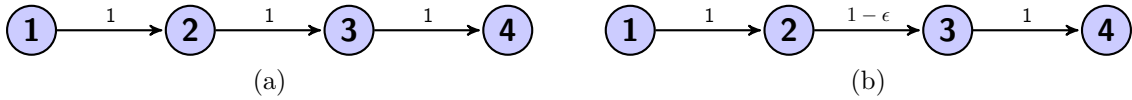


Figure 4: A line network where asset is passed successively from agent to agent until the end-of-line agent

**Example 3.4.** *In figure 4(a), we have a line network, where one unit of asset is passed successively from agents 1 to 4. In this configuration, agent 1 could be a principal and agent 4 an entrepreneur who holds the only unit of asset internal to the network. Agents 2 and 3 are frictionless financial intermediaries in this scenario. The network debt is*

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

*The network debt is concentrated at the end-of-line agent, agent 4; his default would be catastrophic for the network.*

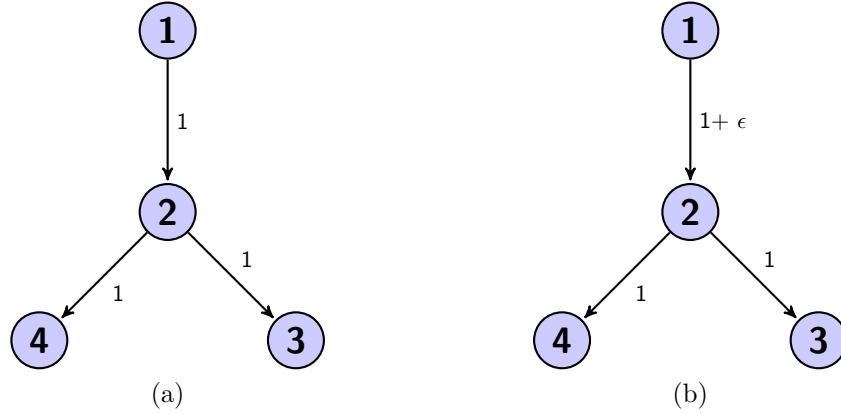


Figure 5: A two-level network where agent 2 serves as an intermediary for two entrepreneurs

**Example 3.5.** *Figure 5(a) depicts a two-level network, with adjacency matrix*

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

*and network debt*

$$\begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

*This may depict a network where agents 1 and 2 have one unit of endowment each. Agent 1 deposits his one-unit asset in agent 2 who then deposits one unit each in agents 3 and 4. Similar to the line network, the agents who are the “sinks” (i.e. have no inter-agent assets) carry non-zero network debt.*

**Change of network debt with respect to liability structure** The above cases give initial indication that network debt reflects the liability structure. Next we inspect how network debt behaves when the liability structures is perturbed in the same examples.

### *Ring network*

Suppose now agent 1 is liable to agent 2 for  $\epsilon$  additional units of debt (see Figure 2(b)). Direct calculation shows network debt after this perturbation is (up to normalization)

$$\begin{bmatrix} \lambda^3 \\ 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

where  $\lambda = (1 + \epsilon)^{\frac{1}{4}} > 1$  is the corresponding eigenvalue. As one who incurred the additional debt, agent 1's network debt gets the most increase. In relative terms, agent 1's network debt increased by a factor of  $\lambda^3$ . The propagation effects on the other agents is determined by the degree of connectedness to agent 1 as a debtor. The second highest increase is suffered by agent 1's immediate debtor, agent 4, with successively less increases for agents 3 and 2.

Similarly, replacing  $\epsilon$  by  $-\epsilon$  gives the opposite result. In that case, after agent 1 reduces his own debt, not only does he now have the lowest network debt, the other agent's network debts also decrease in the same order as their degree of connectedness to agent 1. The agent who derives most decrease in network debt from this is agent 4. In both cases, the agent who is most affected by change in agent 1's liability is agent 4, agent 1's immediate debtor.

### *Line network*

The line network is the limit of smooth perturbations of the ring network. As the liability of agent 1 to agent 2 decreases by  $0 < \epsilon < 1$ , the resulting network debt

$$\begin{bmatrix} \lambda^3 \\ 1 \\ \lambda \\ \lambda^2 \end{bmatrix},$$

with  $\lambda = (1 - \epsilon)^{\frac{1}{4}} < 1$ , changes smoothly with respect to  $\epsilon$ . The limit of network debt as  $\epsilon \rightarrow 1$  is

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

which is exactly the network debt of the line network. In other words, network debt reflects smooth perturbations well. The asymptotic limit of the network debt is the network debt of the limit network as the liability structure changes smoothly. Furthermore, once the line network configuration is achieved, replacing one of the edge weights by  $1 - \epsilon$  does not change the relative network debt. Once an agent achieves a network debt of 0, further decreasing his liability does not change his network debt.

### *Two-level network*

In the two-level network of Figure 5(b), where agent 2 borrows slightly more from agent 1, the default implications for each agent remain the same: As in the unperturbed configuration, agent 2's default necessarily implies that agents 3 or 4 have defaulted. Consistent with this, the resulting network debt distribution is the same. This is a special case of two general facts: if an agent incurs additional debts from other agents *with zero network debt*, the network debt of the entire network remains the same. Borrowing more from someone who is network debt-free causes no penalties, which can be seen informally from the definition. Furthermore, the network debt distribution of the entire network remains the same. This is true in general and stated in Theorem 4.6. Also, this is a case where an agent has a network debt of zero after incurring additional liabilities. It turns out that his prior network debt was already zero. This is a special case of Theorem 4.5.

### *Star network*

The star configuration can be used to demonstrate the general Definition 2. In the two single-creditor star networks shown in Figure 6, the network debt distributions are

$$\begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ \frac{1}{1+3+5} \\ \frac{3}{1+3+5} \\ \frac{5}{1+3+5} \end{bmatrix},$$

respectively. In this star configuration with single creditor,  $d_i$  of a non-central agent is

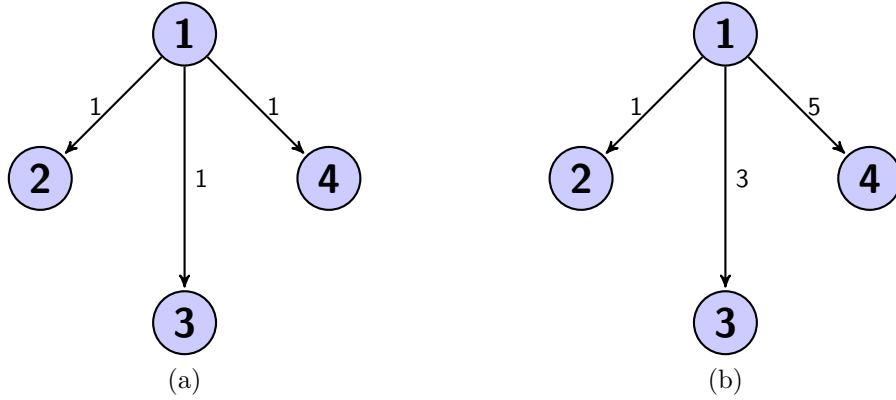


Figure 6: Star networks with different liability structures

exactly agent  $i$ 's proportion of total liabilities in the network,

$$d_i = \frac{\sum_j a_{ij}}{\sum_{kj} a_{kj}}.$$

In general, if  $\sum_j a_{i'j}$  is fixed for  $i \neq i'$ ,  $d_i$  bears a monotonic relationship with  $\frac{\sum_j a_{ij}}{\sum_{kj} a_{kj}}$ .

## 4 General Results

The adjacency matrix  $A = [a_{ij}]$  of a network is (element-wise) non-negative. The well-definedness of (relative) network debt comes from classical Perron-Frobenius theory applied to  $A$ . We state the necessary facts and refer the reader to Minc (1988) for general background.

**Definition 3.** *The adjacency matrix  $A$  is said to be irreducible if, in the corresponding network is strongly connected as a directed graph, i.e. there exists a directed path from  $i$  to  $j$  for all vertices  $i \neq j$ .*

For any non-negative  $A$ , the perturbation  $A + \eta J$  where  $\eta > 0$  and  $J$  is irreducible, is irreducible, i.e. any network can be made strongly connected after an arbitrarily small perturbation.

**Theorem 4.1.** *(Perron-Frobenius) An irreducible non-negative matrix  $A$  has a (unique) real positive eigenvalue  $\lambda$  such that*

$$\lambda \geq |\sigma|$$

*for any other eigenvalue  $\sigma$  of  $A$ . Furthermore, the corresponding eigenvector is unique up to a scalar multiple and can taken to be element-wise positive.*

The maximum positive eigenvalue  $\lambda$  is called the *maximal eigenvalue* of  $A$ . The associated positive eigenvector, the *maximal eigenvector*, is the relative network debt defined in Definition 1. Therefore the notion of network debt is well-defined for strongly connected networks, and, after passing through a limiting procedure, well-defined in general. We first record the immediate fact that Axiom A1 is satisfied.

**Theorem 4.2.** *If agent  $i$  has zero liabilities, then his network debt is 0.*

The converse of Theorem 4.2 is not true, as shown by the perturbed line network example. Additional liability incurred by a financial intermediary does not change his already-zero network debt.

For the examples given in Section 3, change in network debt reflects change in the liability structure. We now establish this formally. Theorem 4.3 answers the following question: if agent  $i$  increases its liabilities, how does this affect the network debt?

**Theorem 4.3.** *In a strongly-connected network of financial liabilities, suppose agent  $i$  increases his liabilities. Let  $d$  and  $d'$  be the network debt distributions before and after the increase in agent  $i$ 's liability profile respectively, then*

$$\log d'_i - \log d_i > \log d'_j - \log d_j, \quad \forall j \neq i.$$

In other words, Axiom A2 holds in relative, or geometric, terms. As agent  $i$  increases his liabilities, his network debt increases the most compared to the rest of the network, in relative terms. In the ring network example, the relative increase in network debt of agent  $j$  is determined by his degree of connectedness as a debtor to agent  $i$ .

Our next theorem says that if, after increasing his liabilities within the network, agent  $i$  is found to have zero network debt, then it already has zero network debt prior to increase. Furthermore, the network debt distribution is unaffected by the additional borrowing. This was demonstrated the two-level network in Section 3.

**Theorem 4.4.** *In the same scenario and same notion as Theorem 4.3, if  $d'_i = 0$ , then  $d'$  and  $d$  are the same.*

In the two-level network, we also saw that if an agent borrows more from a zero network debt creditor, the network debt, for the entire network, is unaffected. This is true in general:

**Theorem 4.5.** *In the same scenario and same notion as above, if the additional liabilities incurred by agent  $i$  is to agents  $j$  such that  $d_j = 0$ , then  $d'$  and  $d$  are the same.*

As shown by the ring network example, the network debt of other agents should decrease smoothly with respect to the increased amount of agent  $i$ 's liability. There are no abrupt jumps in network debts as agent  $i$ 's liability profile varies.

**Theorem 4.6.** *Let  $\Delta r$  be a vector with non-negative entries. Consider the perturbations of the network where agent  $i$  increase his liabilities by  $\eta \cdot \Delta r$  with network debt distribution  $d(\eta)$ . Then there exists  $\delta > 0$  such that, on  $[0, \delta)$ ,  $d_j(\eta)$  is a decreasing differentiable function of  $\eta$ , for  $j \neq i$ .*

Theorem 4.7 is the extension of Theorem 4.3 to the case where more than one agent increases their liabilities.

**Theorem 4.7.** *In a strongly connected network, if a group of agents  $i_1, \dots, i_m$  increase their liabilities, then*

$$\forall j \notin \{i_1, \dots, i_m\}, \log d'_j - \log d_j < \max_{i \in \{i_1, \dots, i_m\}} \log d'_i - \log d_i.$$

As in the single-agent perturbation case, the geometric increase in network debt of an agent that reduced its liabilities is dominated by (the maximum) of those that did not.

Finally, consider the case where one agent becomes more leveraged in the network and a second one less so:

**Theorem 4.8.** *Consider a strongly connected network where agent  $i$  increases his liabilities and agent  $j$  decreases hers. Then for all  $k$ ,*

$$(i) \text{ If } \lambda \leq \lambda', \log d'_k - \log d_k \leq \log d'_i - \log d_i.$$

$$(ii) \text{ If } \lambda \geq \lambda', \log d'_k - \log d_k \geq \log d'_j - \log d_j.$$

$$(iii) \text{ If } \lambda = \lambda', \log d'_j - \log d_j \leq \log d'_k - \log d_k \leq \log d'_i - \log d_i.$$

When the maximal eigenvalue of the network is unaffected by perturbation (case (iii)), the conclusion is a natural one: when change is measured in relative terms,

$$\text{network debt of } j \leq \text{network debt of } k \notin \{i, j\} \leq \text{network debt of } i.$$

We demonstrate this using the ring network. Suppose now agent 1 incurs  $\epsilon$  more liabilities while agent 3 reduced her liabilities by  $\eta$ , as shown in Figure 7. These two changes have

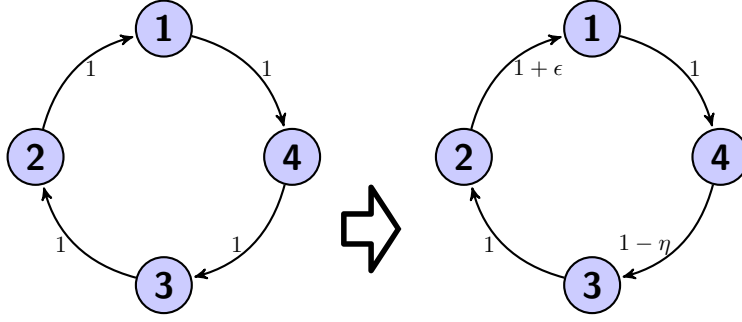


Figure 7: A simultaneous perturbation of two agents' liabilities in the ring network configuration

competing effects on the network. Agent 1's additional liability causes his network debt to increase, with attendant propagation affects downstream to agents for whom he is a creditor. Agent 3's reduction has the opposite effect. If  $\epsilon$  is large relative to  $\eta$ , then the maximal eigenvalue increases. This is case (i) and the conclusion of the theorem focuses on the agent who borrowed more, agent 1. Conversely, if  $\epsilon$  is small relative to  $\eta$ , this results in case (ii) and the theorem's conclusion focuses on the agent who reduced her liabilities, agent 3. The adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 + \epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 - \eta \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

with maximal eigenvalue  $(1 + \epsilon)^{\frac{1}{4}}(1 - \eta)^{\frac{1}{4}}$ . The network debt is, up to normalization,

$$\begin{bmatrix} (1 + \epsilon)^{\frac{1}{4}}(1 - \eta)^{\frac{1}{4}} \\ \frac{(1 - \eta)^{\frac{1}{2}}}{(1 + \epsilon)^{\frac{1}{2}}} \\ \frac{(1 - \eta)^{\frac{3}{4}}}{(1 + \epsilon)^{\frac{1}{4}}} \\ 1 \end{bmatrix}.$$

If  $(1 + \epsilon)(1 - \eta) > 1$ , then agent 1's network debt experiences the most relative increase. Although Theorem 4.8 does not conclude this in general, in this particular case agent 3's network debt decreases, in relative terms. Conversely, if  $(1 + \epsilon)(1 - \eta) < 1$ , agent 3's reduction effect dominates agent 1's additional borrowing, which causes the maximal eigenvalue to decrease.

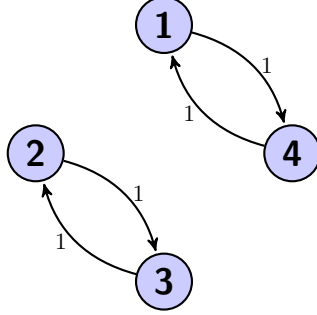


Figure 8: This network is “network debt equivalent” to the ring network

## 5 Limitation

It should be pointed out that our proposed network debt distribution does not “see” components in a network. For example, the network in Figure 8 also has network debt distribution  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ , same as the ring network of Figure 2(a). It has two connected components, while the ring network has one. One default here destroys half (one component) of the network, instead of the entire network, although agents are still homogeneous. Therefore, to fully exploit this notion, it should be accompanied by cluster analysis that separates the network into components or, more generally, weakly connected components.

## 6 Conclusion

Taking a perspective complementary to the existing literature on network liability structures, we have proposed a simple way to measure the severity of agents’ defaults. Without modelling default explicitly, network debt nonetheless takes into account the liability structure. The accompanying general results show that network debt varies with respect to the liability structure in the expected way. We submit that this makes our notion a credible one that lends itself to applications. It offers a simple and computable way of locating vulnerable nodes within an arbitrary network.

Informally, it is natural to suggest that this severity should also correspond to the degree to which an agent is network-leveraged. In other words, network debt, which after normalization is a probability distribution on agents, should have a *cardinal* interpretation in addition to the *ordinal* one we have adopted for the most part. At this point, however, we hesitate to venture such an interpretation beyond what we have been able to prove. Putting the cardinal interpretation on firmer ground is a direction of future work. Another aspect to be explored is marginal network debt under perturbation.

This section contains the proofs. We first fix notation:  $\mathbb{R}_+^{m_1 \times m_2}$  denotes the set of  $m_1 \times m_2$

matrices with non-negative entries and  $(\cdot)^T$  is the transpose operation on matrices. Whenever applicable,  $A = [a_{ij}]$  and  $A' = [a'_{ij}]$  will denote the adjacency matrices of a network before and after a perturbation with respective maximal eigenvalues  $\lambda$  and  $\lambda'$ . For two  $m_1 \times m_2$  matrices  $A$  and  $B$ ,  $A - B \geq 0$  means  $A - B \in \mathbb{R}_+^{m_1 \times m_2}$ .

The following fact will be used repeatedly: Let  $A, B \in \mathbb{R}_+^{n \times n}$  with maximal eigenvalues  $\lambda_A$  and  $\lambda_B$  respectively. Then  $B \leq A$  implies  $\lambda_B \leq \lambda$  with strict inequality if  $A$  is irreducible.

*Proof of Theorem 4.3.* Suppose there is a  $j \neq i$  such that  $\frac{d'_j}{d_j} > \frac{d'_k}{d_k}$  for all  $k \neq j$ . Let  $r_j \in \mathbb{R}_+^{1 \times n}$  be the  $j$ -th rows of  $A$  and  $A'$  (only the  $i$ -th rows of  $A$  and  $A'$  differ). Then

$$\frac{d'_j}{d_j} = \frac{\lambda r_j d'}{\lambda' r_j d} < \frac{r_j d'}{r_j d} = \frac{\sum_{k=1}^n a_{jk} \frac{d'_k}{d_k} d_k}{\sum_{k=1}^n a_{jk} d'_k} < \frac{d'_j}{d_j},$$

which is impossible. □

*Proof of Theorem 4.4.* Let  $x \xrightarrow{P} Px$  denote the linear map on  $\mathbb{R}^n$  where  $Px$  is the vector obtained by setting the  $i$ -th entry  $x_i$  of  $x$  to 0. The assumptions  $A'd' = \lambda'd'$  and  $d_i = 0$  imply

$$PAPd' = \lambda'Pd'.$$

This implies the maximal eigenvalue  $\lambda''$  of  $PAP$  is  $\geq \lambda'$ . But  $\lambda'' \leq \lambda$ . So  $\lambda \geq \lambda' \geq \lambda$ , i.e.  $\lambda = \lambda'$ . So

$$A'd' = \lambda d'$$

Since  $A, A', A - A' \geq 0$ , we have  $Ad' = \lambda d'$ . By the uniqueness of maximal eigenvector up to scalar multiples, the theorem holds. □

*Proof of Theorem 4.5.* Let  $\Delta r \in \mathbb{R}_+^n$  be the additional debt incurred by agent  $i$ . By assumption,  $(\Delta r)^T d = 0$ . If we normalize so that  $d_i = d'_i = 1$ , then  $d \geq d'$  by Theorem 4.3 and a continuity argument. So  $(\Delta r)^T d' = 0$  and,

$$Ad' - \lambda'd' = e_i(\Delta r)^T d' = 0$$

i.e.  $Ad' = \lambda'd'$ . So  $\lambda \geq \lambda' \geq \lambda$ . Same argument in the previous proof now shows that  $d$  and  $d'$  must be the same. □

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<sup>5</sup>We emphasize that  $\geq 0$  often denotes positive semi-definiteness, which is not the case in our notation.

*Proof of Theorem 4.6.* This follows from standard spectral perturbation theory.  $\square$

**Lemma A.1.** *If  $T \in \mathbb{R}_+^{n \times n}$  has maximal eigenvalue  $\lambda$  and  $\alpha > \lambda$ , then  $\alpha - T$  is invertible and  $(\alpha - T)^{-1} \geq 0$ .*

*Proof.* Since  $\mu$  is an eigenvalue of  $T$  if and only if  $\alpha - \mu$  is an eigenvalue of  $\alpha - T$ ,  $\alpha - T$  is invertible. Without loss of generality, assume  $\alpha = 1$ , then the hypothesis  $1 > \lambda$  implies the series

$$I + T + T^2 + \dots$$

converges (entry-wise); its limit is necessarily  $(1 - T)^{-1}$ . Since  $T \geq 0$ , it is clear that  $(1 - T)^{-1} \geq 0$ .  $\square$

**Lemma A.2.** *If invertible matrices  $T$  and  $S \in \mathbb{R}^{n \times n}$  are such that  $T - S \geq 0$ ,  $T^{-1}$ , and  $S^{-1} \geq 0$ , then  $S^{-1} \geq T^{-1}$ .*

*Proof.* We have the general algebraic expression  $S^{-1} - T^{-1} = S^{-1}(T - S)T^{-1}$ . Since all three matrices on the right-hand side are nonnegative, so is their product.  $\square$

*Proof of Theorem 4.8.* (i) Normalize so that  $d_i = d'_i = 1$ . From  $Ad = \lambda d$ , we have

$$Pd = (\lambda - PAP)^{-1}PAe_i$$

where  $P$  is the projection map from the proof of Theorem 4.4. and  $e_i$  is the  $i$ -th standard basis vector. Similarly,

$$Pd' = (\lambda' - C)^{-1}w'$$

where  $0 \leq C \leq PAP$  and  $0 \leq w' \leq PAe_i$ , by the hypothesis that  $\lambda' \geq \lambda$  and the normalization assumption on the  $i$ -th entry of  $d$  and  $d'$ . By Lemma A.1.,  $\lambda' - C$  and  $\lambda - PAP$  are invertible with nonnegative inverses. Since  $\lambda' \geq \lambda$ ,  $\lambda' - C \geq \lambda - PAP$ . Lemma A.2 implies

$$(\lambda' - C)^{-1} \leq (\lambda - PAP)^{-1}.$$

So

$$Pd' = (\lambda' - C)^{-1}w' \leq Pd = (\lambda - PAP)^{-1}PAe_i,$$

which proves the claim. The proof for (ii) is analogous and (iii) follows from (i) and (ii).  $\square$

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